

A COMPLETE MONOTONICITY RESULT INVOLVING THE q -POLYGAMMA FUNCTIONS

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ABSTRACT. We present some completely monotonic functions involving the q -polygamma functions, our result generalizes some known results.

1. INTRODUCTION

For a positive real number x and $q \neq 1$, the q -gamma function is given by

$$\Gamma_q(x) = \begin{cases} (1-q)^{1-x} \prod_{n=0}^{\infty} \frac{1-q^{n+1}}{1-q^{n+x}}, & 0 < q < 1; \\ (q-1)^{1-x} q^{\frac{1}{2}x(x-1)} \prod_{n=0}^{\infty} \frac{1-q^{-(n+1)}}{1-q^{-(n+x)}}, & q > 1. \end{cases}$$

Note that [2, (1.4)] the limit of $\Gamma_q(x)$ as $q \rightarrow 1$ yields the well-known Euler's gamma function:

$$\lim_{q \rightarrow 1} \Gamma_q(x) = \Gamma(x) = \int_0^{\infty} t^x e^{-t} \frac{dt}{t}.$$

Recall that a function $f(x)$ is said to be completely monotonic on (a, b) if it has derivatives of all orders and $(-1)^k f^{(k)}(x) \geq 0, x \in (a, b), k \geq 0$. There exists an extensive and rich literature on inequalities for the gamma and q -gamma functions of positive real numbers. Many of these inequalities follow from the monotonicity properties of functions which are closely related to Γ (resp. Γ_q) and its logarithmic derivative ψ (resp. ψ_q) as ψ' and ψ'_q are completely monotonic functions on $(0, +\infty)$. The derivatives ψ'_q, ψ''_q, \dots are called the q -polygamma functions.

For positive integers r, m, n, s , we denote

$$(1.1) \quad \alpha_{r,m,n,s} = \frac{(m-1)!(n-1)!}{(r-1)!(s-1)!}; \quad \alpha_{r,m,n,0} = \frac{(m-1)!(n-1)!}{(r-1)!}; \quad \beta_{r,m,n,s} = \frac{m!n!}{r!s!}.$$

For integers $r \geq m \geq n \geq s \geq 0$ and any real number t , we define

$$(1.2) \quad F_{r,m,n,s}(x; t) = (-1)^{m+n} \psi^{(m)}(x) \psi^{(n)}(x) - t(-1)^{r+s} \psi^{(r)}(x) \psi^{(s)}(x),$$

where we set $\psi^{(0)}(x) = -1$ for convenience.

In [7, Theorem 4.1], it is shown that when $m+n = r+s$, the function $F_{r,m,n,s}(x; \alpha_{r,m,n,s})$ is completely monotonic on $(0, +\infty)$, while $-F_{r,m,n,s}(x; \beta_{r,m,n,s})$ is also completely monotonic on $(0, +\infty)$ when $s > 0$. This gives a generalization of a result of Alzer and Wells [3, Theorem 2.1], which asserts that for $n \geq 2$, the function $F_{n+1,n,n,n-1}(x; t)$ is strictly completely monotonic on $(0, +\infty)$ if and only if $t \leq (n-1)/n$ and $-F_{n+1,n,n,n-1}(x; t)$ is strictly completely monotonic on $(0, +\infty)$ if and only if $t \geq n/(n+1)$. Following the methods in the proof of [3, Theorem 2.1], it is easy to show that the numbers $\alpha_{r,m,n,s}, \beta_{r,m,n,s}$ are best possible (see also [6]).

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A special case of [7, Theorem 4.1] with $m = n = 1, r = 2, s = 0$ implies that the function $(\psi'(x))^2 + \psi''(x)$ is completely monotonic on $(0, \infty)$. In particular, it implies that

$$(1.3) \quad (\psi'(x))^2 + \psi''(x) \geq 0, \quad x > 0.$$

a result established in the proof of [1, (4.39)](with strict inequality).

We may regard the gamma function as a q -gamma function with $q = 1$ as $\lim_{q \rightarrow 1} \psi_q(x) = \psi(x)$ (see [8]). In this manner, many completely monotonic functions involving $\Gamma_q(x)$ and $\psi_q(x)$ are inspired by their analogues involving $\Gamma(x)$ and $\psi(x)$.

When $q > 1$, the q -analogue of inequality (1.3) is given in [2, Lemma 4.6]:

$$(1.4) \quad (\psi'_q(x))^2 + \psi''_q(x) > 0, \quad x > 0.$$

In [9, Theorem 1.1], it is shown that the function given in (1.4) is completely monotonic for $q > 1$ on $(0, \infty)$.

In [4, Theorem 2.1], it is further shown that the function

$$(1.5) \quad (\psi'_q(x))^2 + \psi''_q(x) - \ln q \cdot \psi'_q(x)$$

is completely monotonic for $q > 0$ on $(0, \infty)$.

We note that inequality (1.3) follows from the limiting case $c \rightarrow 0^+$ of the following inequalities established in the proof of [5, Theorem 1.1]:

$$(1.6) \quad \frac{1}{c} \left(\psi(x+c) - \psi(x) \right)^2 > \psi'(x) - \psi'(x+c), \quad 0 < c < 1, x > 0.$$

More generally, one may replace the derivatives in the expression of $F_{r,m,n,s}(x; t)$ by finite differences and study the complete monotonicity property of the resulting function and this is done in [6].

A q -analogue for inequality (1.6) is given in [7, Theorem 4.2], which asserts that for fixed $0 < q < 1$ and $0 < c < 1$,

$$\frac{1-q}{1-q^c} \left(\psi_q(x+c) - \psi_q(x) \right)^2 > q^x (\psi'_q(x) - \psi'_q(x+c)) > \left(\psi_q(x+c) - \psi_q(x) \right)^2, \quad x > 0,$$

with the above inequalities reversed when $c > 1$.

Motivated by the discussions above, it is natural to replace the derivatives in (1.5) by finite differences to study the complete monotonicity property of the modified function. It is our goal in this paper to study a more general case, the complete monotonicity property of a q -analogue of the function defined in (1.2).

For a given function $f(x)$, any real number $c \neq 0$, we denote

$$\Delta f(x; c) = \frac{f(x+c) - f(x)}{c}.$$

For integers $r \geq m \geq n \geq s \geq 0$, real numbers $q > 0, q \neq 1, c \neq 0$, we define

$$\begin{aligned} & F_{r,m,n,s}(x; q, c) \\ &= (-1)^{m+n} \Delta \psi_q^{(m-1)}(x; c) \Delta \psi_q^{(n-1)}(x; c) - \alpha_{r,m,n,s} (-1)^{r+s} \Delta \psi_q^{(r-1)}(x; c) \Delta \psi_q^{(s-1)}(x; c), \end{aligned}$$

where $\alpha_{r,m,n,s}$ is given in (1.1) and we set $\psi_q^{(0)}(x) = \psi_q(x), \psi_q^{(-1)}(x) = -x$ for convenience.

We further define for integers $m \geq 1$, real numbers $q > 0, q \neq 1, c \neq 0$,

$$G_m(x; q, c) = m F_{m+1,m,1,0}(x; q, c) - (-1)^{m+1} d_{m,q} \ln q \cdot \Delta \psi_q^{(m-1)}(x; c),$$

where

$$(1.7) \quad d_{m,q} = \begin{cases} m-1, & 0 < q < 1, m \geq 2, \\ 1, & q > 1 \quad \text{or} \quad 0 < q < 1, m = 1. \end{cases}$$

Our results are the following:

Theorem 1.1. *Let $q > 0, q \neq 1$ and $0 < c < 1$ be fixed. Let m be any fixed positive integer. The function $G_m(x; q, c)$ is completely monotonic on $(0, \infty)$. Moreover, when $m = 1, 2$, the function $-G_m(x; q, c)$ is completely monotonic on $(0, \infty)$ when $c > 1$.*

Theorem 1.2. *Let $q > 0, q \neq 1$ and $0 < c < 1$ be fixed. Let $r \geq m \geq n \geq s$ be positive integers satisfying $r + s = m + n$. When $s = 1$, the function $F_{r,m,n,1}(x; q, c)$ is completely monotonic on $(0, \infty)$ for $0 < q < 1$. When $s = 2$, the function $F_{r,m,n,2}(x; q, c)$ is completely monotonic on $(0, \infty)$ for $q > 0, q \neq 1$. When $s = 3$, the function $F_{r,m,n,3}(x; q, c)$ is completely monotonic on $(0, \infty)$ for $q > 0, q \neq 1$.*

The $c \rightarrow 0^+$ analogues of the results in Theorem 1.1 and 1.2 can be similarly established using the methods in the proofs of these theorems. Therefore we only state these results below and leave their proofs to the reader.

Corollary 1.1. *Let $q > 0, q \neq 1$ be fixed. Let m be any fixed positive integer. Then the function $m(-1)^{m+1}\psi'_q(x)\psi_q^{(m)}(x) + (-1)^{m+1}\psi_q^{(m+1)}(x) - (-1)^{m+1}d_{m,q} \ln q \cdot \psi_q^{(m)}(x)$ is completely monotonic on $(0, \infty)$, where $d_{m,q}$ is given in (1.7).*

Corollary 1.2. *Let $q > 0, q \neq 1$ be fixed. Let $r \geq m \geq n \geq s \geq 0$ be positive integers satisfying $r + s = m + n$. Let $\alpha_{r,m,n,s}$ be given in (1.1). The function $(-1)^{m+n}\psi_q^{(m)}(x)\psi_q^{(n)}(x) - \alpha_{r,m,n,s}(-1)^{r+s}\psi_q^{(r)}(x)\psi_q^{(s)}(x)$ is completely monotonic on $(0, \infty)$ when $0 < q < 1, s = 1$ or $q > 0, s = 2$ or $q > 0, s = 3, n = 4$.*

2. LEMMAS

For integers $r \geq m \geq n \geq s \geq 1, r + s = m + n$, we define two sequences $\mathbf{a}(m, n), \mathbf{b}(r, m, n, s)$ such that for $k \geq 1$,

$$\mathbf{a}(m, n)_k = \sum_{j=1}^k j^{m-1}(k-j)^{n-1}, \quad \mathbf{b}(r, m, n, s)_k = \alpha_{r,m,n,s} \sum_{j=1}^k (k-j)^{r-1} j^{s-1},$$

where $\alpha_{r,m,n,s}$ is defined in (1.1) and we define $0^0 = 1$.

Our lemmas are concerned with the following inequality for $k \geq 1$:

$$(2.1) \quad \mathbf{a}(m, n)_k \geq \mathbf{b}(r, m, n, s)_k.$$

Lemma 2.1. *Inequality (2.1) holds for integers $k \geq 1, r \geq m \geq n \geq 1, r + 1 = m + n$.*

Proof. For a given sequence $\mathbf{x} = (x_1, x_2, \dots)$, we define $\Delta \mathbf{x}$ to be the sequence satisfying $(\Delta \mathbf{x})_k = x_{k+1} - x_k, k \geq 1$ and we define $\Delta^{(i)} \mathbf{x} = \Delta(\Delta^{(i-1)} \mathbf{x})$ for all $i \geq 2$.

Using slightly different notations by taking integers $s \geq 0, r \geq s, t \geq s$, we see that inequality (2.1) corresponds to the following inequality for $s = 0$:

$$(2.2) \quad \mathbf{a}(r+1, t+1)_k \geq \mathbf{b}(r+t-s+1, r+1, t+1, s+1)_k, \quad k \geq 1.$$

It is ready to see that inequality (2.2) is a consequence of the following inequalities:

$$(2.3) \quad (\Delta^{(s+1)} \mathbf{a}(r+1, t+1))_k \geq (\Delta^{(s+1)} \mathbf{b}(r+t-s+1, r+1, t+1, s+1))_k, \quad k \geq 1.$$

$$(2.4) \quad (\Delta^{(i)} \mathbf{a}(r+1, t+1))_1 \geq (\Delta^{(i)} \mathbf{b}(r+t-s+1, r+1, t+1, s+1))_1, \quad 1 \leq i \leq s.$$

When $s = 0$, inequality (2.4) holds trivially so it remains to prove inequality (2.3). We may now assume $r > 1, t \geq 1$. When $t = 1$, inequality (2.3) becomes the following easily verified inequality:

$$\sum_{j=1}^k j^r \geq \frac{1}{r+1} k^{r+1}.$$

We then deduce that inequality (2.1) is valid when $s = 0, n = 2$.

Now, we use induction on $k \geq 1$ to prove inequality (2.3) for all $r > 1, t \geq 1$. The case $k = 1$ is easy verified. Now assuming that inequality (2.3) is valid for all $k \geq 1, 1 \leq t \leq T - 1$. When $t = T$, we apply the binomial expansion to see that inequality (2.3) becomes

$$(2.5) \quad \sum_{i=0}^{T-1} \binom{T}{i} \sum_{j=1}^k j^r (k-j)^i \geq \frac{r!T!}{(r+T)!} k^{r+T}.$$

Applying the induction assumption, we see that

$$\binom{T}{i} \sum_{j=1}^k j^r (k-j)^i \geq \binom{T}{i} \cdot \frac{r!i!}{(r+i)!} \sum_{j=1}^{k-1} j^{r+i}, \quad 1 \leq i \leq T-1.$$

Using this in (2.5), we see that it remains to show that (with empty sums being 0)

$$S_k := \sum_{i=1}^{T-1} \binom{r+T}{r+i} \sum_{j=1}^{k-1} j^{r+i} + \binom{r+T}{r} \sum_{j=1}^k j^r \geq k^{r+T}.$$

As the above inequality is valid when $k = 1$, it suffices to show that

$$S_{k+1} - S_k \geq (k+1)^{r+T} - k^{r+T}.$$

The above inequality simplifies to be

$$\sum_{i=1}^{T-1} \binom{r+T}{r+i} k^{r+i} + \binom{r+T}{r} (k+1)^r \geq (k+1)^{r+T} - k^{r+T} = \sum_{i=0}^{r+T-1} \binom{r+T}{i} k^i.$$

We can further recast the above inequality as

$$\binom{r+T}{r} (k+1)^r \geq \sum_{i=0}^r \binom{r+T}{i} k^i.$$

As the above inequality is easily verified by first applying the binomial expansion to $(k+1)^r$ and then to compare the corresponding coefficients of $k^i, 0 \leq i \leq T$ on both sides, this implies that inequality (2.3) is valid for all $r \geq 1, t \geq 1, s = 0$ and hence inequality (2.1) is valid for all $k \geq 1, r \geq m \geq n \geq 1, r+1 = m+n$. \square

Lemma 2.2. *For integers $k \geq 1, r \geq m \geq n \geq s, r+s = m+n$, inequality (2.1) holds when $s = 2$ or when $n = 4, s = 3$.*

Proof. Using the notations in Lemma 2.1, we apply the binomial expansion to see that for integers $T \geq s \geq 1, r \geq s, k \geq 1$,

$$\begin{aligned}
(\Delta \mathbf{a}(r+1, T+1))_k &= \sum_{t=0}^{T-1} \binom{T}{t} \mathbf{a}(r+1, t+1)_k, \\
(\Delta \mathbf{b}(r+T-s+1, r+1, T+1, s+1))_k \\
&= \alpha_{r+T-s+1, r+1, T+1, s+1} \sum_{t=s-r}^{T-1} \binom{r+T-s}{r+t-s} \sum_{j=1}^k j^s (k-j)^{r+t-s} \\
&= \sum_{t=s-r}^{T-1} \binom{T}{t} \mathbf{b}(r+t-s+1, r+1, t+1, s+1)_k \\
&= \sum_{t=s}^{T-1} \binom{T}{t} \mathbf{b}(r+t-s+1, r+1, t+1, s+1)_k \\
&\quad + \sum_{t=s-r}^{s-1} \binom{T}{t} \mathbf{b}(r+t-s+1, r+1, t+1, s+1)_k.
\end{aligned}$$

It follows that for integers $i \geq 1, k \geq 1$, we have (with empty sums being 0)

$$\begin{aligned}
(2.6) \quad (\Delta^{(i+1)} \mathbf{a}(r+1, T+1))_k &= \sum_{t=0}^{T-1} \binom{T}{t} (\Delta^{(i)} \mathbf{a}(r+1, t+1))_k, \\
(\Delta^{(i+1)} \mathbf{b}(r+T-s+1, r+1, T+1, s+1))_k \\
&= \sum_{t=s}^{T-1} \binom{T}{t} (\Delta^{(i)} \mathbf{b}(r+t-s+1, r+1, t+1, s+1))_k \\
&\quad + \sum_{t=s-r}^{s-1} \binom{T}{t} (\Delta^{(i)} \mathbf{b}(r+t-s+1, r+1, t+1, s+1))_k.
\end{aligned}$$

Suppose that for fixed $r \geq s, T \geq s \geq 1$, we can show that for all $k \geq 1$,

$$(2.7) \quad \sum_{t=0}^{s-1} \binom{T}{t} (\Delta^{(s+1)} \mathbf{a}(r+1, t+1))_k \geq \sum_{t=s-r}^{s-1} \binom{T}{t} (\Delta^{(s+1)} \mathbf{b}(r+t-s+1, r+1, t+1, s+1))_k.$$

As we have trivially for any $k \geq 1$,

$$(\Delta^{(s+1)} \mathbf{a}(r+1, s+1))_k = (\Delta^{(s+1)} \mathbf{b}(r+1, r+1, s+1, s+1))_k.$$

It follows that if for any $s \leq t < T$, we have

$$(2.8) \quad (\Delta^{(s+1)} \mathbf{a}(r+1, t+1))_k \geq (\Delta^{(s+1)} \mathbf{b}(r+t-s+1, r+1, t+1, s+1))_k, \quad k \geq 1.$$

Then inequalities (2.6) and (2.7) imply that

$$(\Delta^{(s+2)} \mathbf{a}(r+1, T+1))_k \geq (\Delta^{(s+2)} \mathbf{b}(r+T-s+1, r+1, T+1, s+1))_k, \quad k \geq 1,$$

which in turn implies that inequality (2.8) is valid with $t = T$ there, provided we show that inequality (2.4) is valid for $t = T, i = s+1$.

We then conclude that inequality (2.2) is valid for all $t \geq s \geq 1$ provided that both inequalities (2.7) and (2.4) (for $1 \leq i \leq s+1$) are valid.

To facilitate the calculation of the right-hand side expression in (2.7), we note that

$$\sum_{t=s-r}^{s-1} \binom{T}{t} \mathbf{b}(r+t-s+1, r+1, t+1, s+1)_k = \alpha_{r+T-s+1, r+1, T+1, s+1} \sum_{t=0}^{r-1} \binom{r+T-s}{t} \mathbf{c}(t, s)_k,$$

where for integers $k, s \geq 1, t \geq 0$,

$$\mathbf{c}(t, s)_k = \sum_{j=0}^{k-1} j^t (k-j)^s.$$

It follows that

$$\begin{aligned} & \sum_{t=s-r}^{s-1} \binom{T}{t} (\Delta^{(s+1)} \mathbf{b}(r+t-s+1, r+1, t+1, s+1))_k \\ &= \alpha_{r+T-s+1, r+1, T+1, s+1} \sum_{t=0}^{r-1} \binom{r+T-s}{t} (\Delta^{(s+1)} \mathbf{c}(t, s))_k, \end{aligned}$$

Now, the case $s = 2$ of inequality (2.1) corresponds to the case $s = 1$ of inequality (2.2). In this case, it is readily checked that inequality (2.7) becomes the following inequality, which is seen to be valid by comparing the corresponding coefficients of $(k+1)^t, 0 \leq t \leq r-1$:

$$(k+2)^r - (k+1)^r = \sum_{t=0}^{r-1} \binom{r}{t} (k+1)^t \geq \alpha_{r+T, r+1, T+1, 2} \sum_{t=0}^{r-1} \binom{r+T-1}{t} (k+1)^t.$$

It remains to prove inequality (2.4) for $t \geq s = 1, 1 \leq i \leq 2$. The case $i = 1$ holds trivially and when $i = 2$, inequality (2.4) becomes

$$D(r, t) := 2^r + 2^t - 2 - \frac{r!t!}{(r+t-1)!} \cdot 2^{r+t-1} \geq 0.$$

We may assume that $r \geq t$ and note that the above inequality becomes an identity when $t = 1$. We check that

$$D(r, t+1) - D(r, t) = 2^t - \frac{r!t!}{(r+t)!} \cdot 2^{r+t-1} \cdot (t+2-r).$$

The right-hand side expression above is easily seen to be positive when $r \geq t+2$, while $D(t, t+1) - D(t, t) \geq 0$ and $D(t+1, t+1) - D(t+1, t) \geq 0$ are consequences of the following easily verified inequalities:

$$\frac{(2t+1)!}{(t+1)!t!} \geq \frac{(2t)!}{t!t!} \geq 2^t.$$

This shows that inequality (2.1) is valid for $s = 2$.

Next, the case $s = 3, n = 4$ of inequality (2.1) corresponds to the case $s = 2, t = 3$ of inequality (2.2). As inequality (2.2) holds trivially for $s = t = 2$, by our discussions above, it remains to prove (2.7) for $s = 2, T = 3$ and (2.4) for $s = 2, t = 3, 1 \leq i \leq 3$.

Inequality (2.4) for $s = 2, t = 3, 1 \leq i \leq 3$ are readily checked to be valid, while inequality (2.7) for $s = 2, T = 3$ becomes

$$\begin{aligned} & (k+3)^r - 2(k+2)^r + (k+1)^r + T((k+2)^r - (k+1)^r) \\ & \geq \sum_{t=0}^{r-1} \frac{r!T!}{2 \cdot t!(r+T-2-t)!} ((k+1)^t + (k+2)^t). \end{aligned}$$

When $T = 3$, the above inequality becomes

$$(k+3)^r + 4(k+2)^r + (k+1)^r \geq \frac{3}{r+1}((k+3)^{r+1} - (k+1)^{r+1}).$$

Applying the binomial expansion to write both sides above as sums of powers of $k+1$ and by comparing the corresponding coefficients, we see that it suffices to show for $0 \leq i \leq r-1$,

$$\binom{r}{i}(2^{r-i} + 4) \geq \frac{3}{r+1} \binom{r+1}{i} 2^{r+1-i},$$

which is equivalent to

$$4(r+1-i) \geq (5+i-r)2^{r-i}.$$

One checks easily that the above inequality is valid for $r-4 \leq i \leq r-1$ and holds trivially when $i \leq r-5$. This proves (2.7) for $s = 2, T = 3$ and hence completes the proof for the case $s = 3, n = 4$ of inequality (2.1). \square

3. PROOF OF THEOREM 1.1

We note the following expressions for $\psi_q(x)$, which can be found in [4, (1.3)-(1.4)]:

$$(3.1) \quad \begin{aligned} \psi_q(x) &= -\ln(q-1) + \ln q \left(x - \frac{1}{2} - \sum_{n=1}^{\infty} \frac{q^{-nx}}{1-q^{-n}} \right), \quad q > 1, x > 0, \\ \psi_q(x) &= -\ln(1-q) + \ln q \sum_{n=1}^{\infty} \frac{q^{nx}}{1-q^n}, \quad 0 < q < 1, x > 0. \end{aligned}$$

For two variables x, y , we define $\delta_{xy} = 1$ when $x = y$ and $\delta_{xy} = 0$ otherwise. It follows from the above expressions that

$$(3.2) \quad \begin{aligned} \psi_q(x) &= (x - \frac{3}{2}) \ln q + \psi_{1/q}(x), \\ \psi_q^{(m)}(x) &= \delta_{1m} \ln q + \psi_{1/q}^{(m)}(x), \quad m \geq 1. \end{aligned}$$

We then deduce that $G_m(x; q, c) = G_m(x; 1/q, c)$, hence it suffices to prove Theorem 1.1 by assuming that $q > 1$. Using (3.1), we see that when $q > 1$,

$$G_m(x; q, c) = \frac{(\ln q)^{m+1}}{c^2} \sum_{n=2}^{\infty} q^{-nx} H(m, n, q, c) + (1 - \delta_{1m})(\ln q)^{m+1} q^{-x} \frac{(m-2)(1-q^{-c})}{(1-q^{-1})c},$$

where

$$\begin{aligned} &H(m, n, q, c) \\ &= \sum_{j=1}^{n-1} \frac{(1-q^{-jc})(1-q^{-(n-j)c})m(n-j)^{m-1}}{(1-q^{-j})(1-q^{-(n-j)})} - \frac{c(1-q^{-nc})(n^m - (m-1)n^{m-1} - \delta_{1m})}{1-q^{-n}}. \end{aligned}$$

It suffices to show $H(m, n, q, c) \geq 0$ for all $n \geq 2$. We first show that for any $1 \leq j \leq n-1$, $0 < c < 1$,

$$(3.3) \quad \frac{(1-q^{-jc})(1-q^{-(n-j)c})}{(1-q^{-j})(1-q^{-(n-j)})} \geq \frac{c(1-q^{-nc})}{1-q^{-n}},$$

with the above inequality reversed when $c > 1$. We denote $y = q^{-1}$ so that $0 < y < 1$ and we let

$$h(z; y, c) = \ln \frac{1-y^{zc}}{1-y^z}.$$

It is then easy to see that inequality (3.3) is equivalent to

$$(3.4) \quad h(j; y, c) + h(n - j; y, c) \geq \lim_{t \rightarrow 0^+} (h(t; y, c) + h(n - t; y, c)),$$

with the above inequality reversed when $c > 1$.

Applying [7, Lemma 2.4], we see that (3.4) follows if we can show $h(z; y, c)$ is a concave function of z when $0 < c < 1$ and a convex function of z when $c > 1$. Direct calculation shows that

$$h''(z; y, c) = \frac{(\ln y)^2 y^z}{(1 - y^z)^2} - \frac{(\ln y^c)^2 y^{zc}}{(1 - y^{zc})^2}.$$

As it is easy to check that the function

$$x \mapsto \frac{u(\ln u)^2}{(1 - u)^2}$$

is an increasing function of $0 < u < 1$, it follows readily that $h''(z; y, c) \leq 0$ when $0 < c < 1$ and $h''(z; y, c) \geq 0$ when $c > 1$, so that inequality (3.3) follows.

Now, applying inequality (3.3) in the expression of $H(m, n, q, c)$, we see that the assertion for Theorem 1.1 is valid as long as we can show for integers $n \geq 2, m \geq 1$,

$$(3.5) \quad m \sum_{j=1}^{n-1} (n - j)^{m-1} \geq n^m - (m - 1)n^{m-1} - \delta_{1m},$$

with equality holds when $m = 1, 2$.

It is easy to see that inequality (3.5) becomes equality when $m = 1, 2$ and this proves the assertion of Theorem 1.1 for the case $m = 1, 2$. In fact, as one checks easily $-G_2(x; q, c) = G'_1(x; q, c)$, the assertion of Theorem 1.1 for the case $m = 2$ follows from its assertion for the case $m = 1$.

Now, we assume $m \geq 2$ and we let $r = m - 1$ to recast inequality (3.5) as

$$(3.6) \quad (r + 1) \sum_{i=1}^n i^r \geq (n + 1)^{r+1} - r(n + 1)^r, \quad r \geq 1, \quad n \geq 1.$$

Note that by Hadamard's inequality ([7, Lemma 2.5]), we have

$$\int_n^{n+1} x^r dr \leq \frac{n^r + (n + 1)^r}{2} \leq \frac{r}{r + 1} (n + 1)^r + \frac{1}{r + 1} n^r, \quad n \geq 1.$$

It is easy to see that inequality (3.6) follows from the above inequality and induction (with the case $n = 1$ in (3.6) following from the case $n = 1$ of the above inequality). This completes the proof for Theorem 1.1.

4. PROOF OF THEOREM 1.2

It follows from (3.2) that when $s \geq 2$, we have $F_{r,m,n,s}(x; q, c) = F_{r,m,n,s}(x; 1/q, c)$. Hence, it suffices to prove the assertion of Theorem 1.2 by assuming that $0 < q < 1$. We start by considering the function $F_{r,m,n,s}(x; q, c)$ with $r > m \geq n > s \geq 1, r + s = m + n$. Using (3.1), we see that when $0 < q < 1, s > 0$,

$$\begin{aligned} & F_{r,m,n,s}(x; q, c) \\ &= \frac{(-\ln q)^{m+n}}{c^2} \sum_{k=2}^{\infty} q^{kx} \sum_{j=1}^{k-1} \frac{(1 - q^{jc})(1 - q^{(k-j)c})}{(1 - q^j)(1 - q^{k-j})} (j^{m-1}(k - j)^{n-1} - \alpha_{r,m,n,s} j^{r-1}(k - j)^{s-1}), \end{aligned}$$

We then deduce that in order for $F_{r,m,n,s}(x; q, c)$ to be completely monotonic on $(0, \infty)$, it suffices to show the inner sum in the above expression is non-negative for all $k \geq 2$. We recast the inner

sum above as

$$\sum_{j=1}^{\lfloor k/2 \rfloor} \frac{(1-q^{jc})(1-q^{(k-j)c})}{(1-q^j)(1-q^{k-j})} T_{m,n,r,s}(j; k) \left(1 - \frac{\delta_{j\frac{k}{2}}}{2}\right),$$

where

$$\begin{aligned} T_{m,n,r,s}(j; k) &= j^{m-1}(k-j)^{n-1} + (k-j)^{m-1}j^{n-1} - \alpha_{r,m,n,s}(j^{r-1}(k-j)^{s-1} + (k-j)^{r-1}j^{s-1}) \\ &= j^{r-1}(k-j)^{s-1}a(k/j-1; r-s, m-s, \alpha_{r,m,n,s}), \end{aligned}$$

where for integers $m > n \geq 1$, any fixed constant $0 < c < 1$, the function $t \mapsto a(t; m, n, c)$ is defined as in [7, Lemma 2.7] (we note here that it is easy to verify that $0 < \alpha_{r,m,n,s} < 1$).

It is shown in [7, Lemma 2.7] that $a(t; m, n, c)$ has exactly one root when $t \geq 1$ for any integer $m > n \geq 1$, any fixed constant $0 < c < 1$. Note that $\lim_{t \rightarrow \infty} a(t; m, n, c) = -\infty$, $a(1; m, n, c) > 0$, it follows that there exists an integer $1 \leq j_0 \leq \lfloor k/2 \rfloor$ such that $a(k/j-1; r-s, m-s, \alpha_{r,m,n,s}) < 0$ for $j < j_0$ and $a(k/j-1; r-s, m-s, \alpha_{r,m,n,s}) \geq 0$ for $j_0 \leq j \leq \lfloor k/2 \rfloor$.

If $j_0 = 1$, then the assertion of the theorem holds trivially. Otherwise, note that our argument for inequality (3.3) shows that the sequence

$$\left\{ \frac{(1-q^{jc})(1-q^{(k-j)c})}{(1-q^j)(1-q^{k-j})} \right\}_{j=1}^{\lfloor k/2 \rfloor}$$

is a positive increasing sequence, thus we have

$$\frac{(1-q^c)(1-q^{(k-1)c})}{(1-q)(1-q^{k-1})} T_{m,n,r,s}(1; k) \geq \frac{(1-q^{2c})(1-q^{(k-2)c})}{(1-q^2)(1-q^{k-2})} T_{m,n,r,s}(1; k),$$

so that

$$\sum_{j=1}^2 \frac{(1-q^{jc})(1-q^{(n-j)c})}{(1-q^j)(1-q^{n-j})} T_{m,n,r,s}(j; k) \geq \frac{(1-q^{2c})(1-q^{(n-2)c})}{(1-q^2)(1-q^{n-2})} \sum_{j=1}^2 T_{m,n,r,s}(j; k).$$

Consider the sequence of the partial sums:

$$\left\{ \sum_{j=1}^i T_{m,n,r,s}(j; k) \right\}_{i=1}^{\lfloor k/2 \rfloor - 1}.$$

If there exists an integer $1 \leq i_0 \leq \lfloor k/2 \rfloor - 1$ such that the terms in the above sequence are negative when $i < i_0$ and the term corresponding to $i = i_0$ is non-negative, then we can repeat the above process to see that (note that we must have $i_0 \geq j_0$ here)

$$\begin{aligned} & \sum_{j=1}^{\lfloor k/2 \rfloor} \frac{(1-q^{jc})(1-q^{(n-j)c})}{(1-q^j)(1-q^{n-j})} T_{m,n,r,s}(j; k) \left(1 - \frac{\delta_{j\frac{k}{2}}}{2}\right) \\ & \geq \frac{(1-q^{i_0c})(1-q^{(n-i_0)c})}{(1-q^{i_0})(1-q^{n-i_0})} \sum_{j=1}^{i_0} T_{m,n,r,s}(j; k) \left(1 - \frac{\delta_{j\frac{k}{2}}}{2}\right) \\ & \quad + \sum_{j=i_0+1}^{\lfloor k/2 \rfloor} \frac{(1-q^{jc})(1-q^{(n-j)c})}{(1-q^j)(1-q^{n-j})} T_{m,n,r,s}(j; k) \left(1 - \frac{\delta_{j\frac{k}{2}}}{2}\right) \geq 0. \end{aligned}$$

If no such i_0 exists, then we have

$$\begin{aligned} & \sum_{j=1}^{[k/2]} \frac{(1-q^{jc})(1-q^{(n-j)c})}{(1-q^j)(1-q^{n-j})} T_{m,n,r,s}(j; k) \left(1 - \frac{\delta_{j\frac{k}{2}}}{2}\right) \\ & \geq \frac{(1-q^{i_0c})(1-q^{(n-j)[k/2]})}{(1-q^{[k/2]})(1-q^{n-[k/2]})} \sum_{j=1}^{[k/2]} T_{m,n,r,s}(j; k) \left(1 - \frac{\delta_{j\frac{k}{2}}}{2}\right). \end{aligned}$$

Hence, it remains to show that

$$\sum_{j=1}^{[k/2]} T_{m,n,r,s}(j; k) \left(1 - \frac{\delta_{j\frac{k}{2}}}{2}\right) \geq 0.$$

We can recast the above inequality as inequality (2.1) and the assertion of Theorem 1.2 now follows from Lemmas 2.1-2.2.

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